

FOCK SPACE REPRESENTATIONS AND CRYSTAL BASES FOR $C_n^{(1)}$

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Abstract

We describe the Fock space representations and crystal bases for the affine Kac-Moody Lie algebra of type $C_n^{(1)}$ in terms of coloured Young diagrams.

INTRODUCTION

In [2], the crystals for a highest weight representations of the quantized universal enveloping algebra of $\hat{\mathfrak{sl}}(n)$ are given in terms of coloured Young diagrams using the Fock space representations of this algebra. In [4], [8] and [7], these results are generalized to some representations (those with dominant integral highest weights of level 1) of the affine Lie algebras of types $A_{2n-1}^{(2)}$, $n \geq 3$, $D_n^{(1)}$, $n \geq 4$, $A_{2n}^{(2)}$, $n \geq 2$, $D_{n+1}^{(2)}$, $n \geq 2$, and $B_n^{(1)}$, $n \geq 3$. The new combinatorial objects for these algebras are called Young walls and are built out of cubes and “half-cubes”. Using Young walls to obtain the results for the algebras of type $C_n^{(1)}$ proved to be more difficult. In [5] and [6], the authors obtain the results for \mathfrak{g} of type $C_2^{(1)}$ and in [1] they obtain a description of the crystal base for $C_n^{(1)}$ using Young walls. In this paper, we define the Fock space representations of any fundamental weight for the quantized universal enveloping algebra \mathfrak{g} of type $C_n^{(1)}$, $n \geq 2$, using Young diagrams (two-dimensional combinatorial objects) coloured appropriately - the colouring is different from that for $\hat{\mathfrak{sl}}(n)$. We then obtain a description of the crystal of an irreducible representation with a dominant highest weight in terms of coloured Young diagrams.

PRELIMINARIES

In this section we set up the notation and state some definitions which will be needed in the following sections. Let $I = \{0, 1, \dots, n\}$ and

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -2 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ & & & \ddots & \\ \dots & & -1 & 2 & -1 & 0 \\ \dots & & 0 & -1 & 2 & -2 \\ \dots & & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $(\mathfrak{h} = \bigoplus_{i=0}^n \mathbb{Q}h_i \oplus \mathbb{Q}d, \Pi = \{h_i : i \in I\}, \Pi^\vee = \{\alpha_i : i \in I\})$ be a realization of A (see [3]) and \mathfrak{g} denote the affine Kac-Moody Lie algebra of type $C_n^{(1)}$. Hence we have $\alpha_j(h_i) = a_{ij}$. Let $d \in \mathfrak{h}$ be an element of \mathfrak{h} such that $\alpha_i(d) = \delta_{0,i}$. Define $s_0 = s_n = 2$ and $s_i = 1$ for $i \in I$, $i \neq 0, n$, so we have that $(s_i a_{ij})_{i,j \in I}$ is symmetric.

The **quantized universal enveloping algebra** of \mathfrak{g} , $\mathcal{U}_q(\mathfrak{g})$, is the associative algebra over $\mathbb{Q}(q)$ generated by the elements e_i , f_i , $i \in I$, and q^h , $h \in P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d$, subject to the following relations:

- (1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$, for h and $h' \in P^\vee$,
- (2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$, for $h \in P^\vee$,
- (3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$, for $h \in P^\vee$,
- (4) $e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$, for $i, j \in I$,
- (5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,
- (6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$,

where $q_i = q^{s_i}$, $k_i = q^{s_i h_i}$, $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[0]! = 1$, $[n]_{q_i}! = [n]_{q_i} [n-1]_{q_i} \cdots [1]_{q_i}$ and

$$\begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[n]_{q_i}! [m-n]_{q_i}!}.$$

The **weight lattice** for \mathfrak{g} is defined to be $P := \{\lambda \in \mathfrak{h}^* : \lambda(P^\vee) \subset \mathbb{Z}\}$.

A $\mathcal{U}_q(\mathfrak{g})$ -module M is said to belong to the category \mathcal{O}_{int} if

- (i) $M = \bigoplus_{\lambda \in P} M_\lambda$ where $M_\lambda = \{u \in M : q^h u = q^{\lambda(h)} u \ \forall h \in P^\vee\}$,
- (ii) $\dim(M_\lambda) < \infty$ for all $\lambda \in P$,
- (iii) for each $i \in I$, M is the union of finite dimensional $\mathcal{U}_q(\mathfrak{g}_i)$ -modules where \mathfrak{g}_i is the subalgebra of \mathfrak{g} generated by e_i, f_i, q^{h_i} and q^{-h_i} , and
- (iv) $M = \bigoplus_{\lambda \in F+Q_-} M_\lambda$, where F is a finite subset of P and $Q_- = -\sum_{i \in I} \mathbb{N}\alpha_i$.

The category \mathcal{O}_{int} is semisimple with irreducible objects $\{V(\lambda) : \lambda \in P_+\}$, where $\lambda \in P_+ := \{\lambda \in \mathfrak{h}^* : \lambda(P^\vee) \subset \mathbb{N}\}$.

A weight module, M , satisfying (i) above is said to be a **highest weight module of highest weight** λ if there exists a $u \in M$ such that

- (i) $e_i u = 0$ for all $i \in I$,
- (ii) $q^h u = q^{\lambda(h)} u$ for all $h \in P^\vee$, and
- (iii) $M = \mathcal{U}_q(\mathfrak{g})u$.

Every highest weight module in the category \mathcal{O}_{int} is isomorphic to $V(\lambda)$ for some $\lambda \in P_+$.

For $k \in I$, define $\Lambda_k \in P_+$ by $\Lambda_k(h_j) = \delta_{kj}$, and $\Lambda_k(d) = 0$. In the following sections we will define the Fock space representation for Λ_k . The corresponding module belongs to the category \mathcal{O}_{int} and contains $V(\Lambda_k)$. We will use this to describe $B(\Lambda_k)$ by coloured Young diagrams, where $(L(\Lambda_k), B(\Lambda_k))$ denotes the (lower or upper) crystal base of $V(\Lambda_k)$ (see [9]).

THE FOCK SPACE REPRESENTATIONS FOR \mathfrak{g} OF TYPE $C_n^{(1)}$

Here we modify the definitions in [2] to define the Fock space representations of $\mathcal{U}_q(\mathfrak{g})$.

2.1. Definition. A **Young diagram Y of charge k** , for $k \in I$, is a sequence $\{y_l\}_{l \in \mathbb{N}}$ such that

- (i) $y_l \in \mathbb{Z}$,
- (ii) $y_l \leq y_{l+1}$ for all $l \in \mathbb{N}$, and
- (iii) $y_l = i$ for all $l \gg 0$.

The empty Young diagram of charge k will be denoted by ϕ_k , i.e. $\phi_k = (k, k, \dots)$. Define

$$\mathcal{Y}(\Lambda_k) := \{\mathbf{Y} : \mathbf{Y} \text{ is a Young diagram of charge } k\},$$

and the **Fock space** of weight Λ_k to be

$$\mathcal{F}(\Lambda_k) = \bigoplus_{\mathbf{Y} \in \mathcal{Y}} \mathbb{Q}\mathbf{Y}.$$

We colour the x - y plane as follows: For l and $l' \in \mathbb{Z}$, the “box” $\{(x, y) : l < x \leq l+1, l'-1 < y \leq l'\}$ is coloured i where $i \in I$ and $l+l' \equiv \pm i \pmod{2n}$ (see

(0,0)

	0	1	...	$n-1$	n	$n-1$...	1	0
	1	0	...	$n-2$	$n-1$	n	...	2	1
	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	$n-1$	$n-2$...	0	1	2	...	n	$n-1$
	n	$n-1$...	1	0	1	...	$n-1$	n
	$n-1$	n	...	2	1	0	...	$n-2$	$n-1$
	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
	1	2	...	n	$n-1$	$n-2$...	0	1
	0	1	...	$n-1$	n	$n-1$...	1	0

FIGURE 1. The colouring of the x-y plane.

Figure 1). Then the diagram $Y = \{y_l\}_{l \geq 0}$ is represented in the coloured x - y plane by the coloured region defined by $\{(x, y) : l \leq x \leq l+1, 0 \geq y \geq y_l \text{ for some } l \in \mathbb{N}\}$.

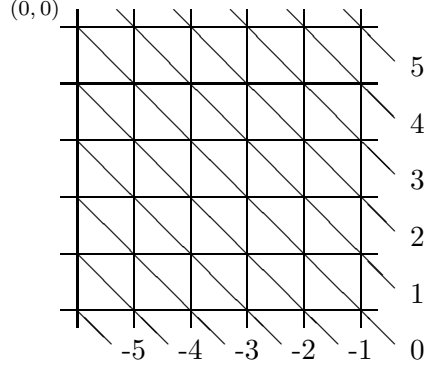
Let $\mathbf{Y} = \{y_l\}_{l \in \mathbb{N}} \in \mathcal{Y}(\Lambda_k)$. If $y_l \neq y_{l+1}$ for some $l \in \mathbb{N}$, \mathbf{Y} is said to have a **concave (convex) corner** at site $(l+1, y_{l+1})$ ($(l+1, y_l)$, resp.). Also, \mathbf{Y} is said to have a **concave corner** at site $(0, y_l)$. For $i \in I$, a corner at site (l, y) is called an **i -coloured corner** if $l + y \equiv \pm i \pmod{2n}$.

2.1.1. **Example.** Let $n = 2$ and $\mathbf{Y} = (-4, -2, -2, -1, -1, 0, 0, \dots)$.

$$\mathbf{Y} = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 1 & 0 \\ \hline 1 & 0 & 1 & & \\ \hline 2 & & & & \\ \hline 1 & & & & \\ \hline \end{array}$$

\mathbf{Y} has a 0-coloured concave corner at sites $(0, -4)$, a 0-coloured convex corner at site $(5, -1)$, 1-coloured concave corners at sites $(0, 5)$ and $(1, -2)$, 1-coloured convex corners at sites $(3, -2)$, and $(1, -4)$, and a 2-coloured concave corner at sites $(3, -1)$.

2.2. We now define an action of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{F}(\Lambda_k)$.

FIGURE 2. The order of $\mathbb{N} \times \mathbb{Z}$.

For $(l, y) \in \mathbb{N} \times \mathbb{Z}$ define linear maps $E_{(l,y)}$, $F_{(l,y)}$, $T_{(l,y)}^{\pm} : \mathcal{F}(\Lambda_k) \rightarrow \mathcal{F}(\Lambda_k)$ as follows, for $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$,

if \mathbf{Y} has a convex corner at site (l, y) , $E_{(l,y)}(\mathbf{Y})$ is the same as \mathbf{Y} with this corner removed; otherwise, $E_{(l,y)}(\mathbf{Y}) = 0$,

if \mathbf{Y} has a concave corner at site (l, y) , $F_{(l,y)}(\mathbf{Y})$ is the same as \mathbf{Y} with a corner added at site $(l+1, y-1)$; otherwise, $F_{(l,y)}(\mathbf{Y}) = 0$,

$$T_{(l,y)}^{\pm}(\mathbf{Y}) = \begin{cases} q_i^{\pm} \mathbf{Y} & \text{if } \mathbf{Y} \text{ has a concave corner at site } (l,y), \\ q_i^{\mp} \mathbf{Y} & \text{if } \mathbf{Y} \text{ has a convex corner at site } (l,y), \\ \mathbf{Y} & \text{otherwise,} \end{cases}$$

where $i \in I$ such that $l + y \equiv \pm i \pmod{2n}$.

Define the order $>$ on $\mathbb{N} \times \mathbb{Z}$ as follows: $(l, y) > (l', y')$ iff $l + y > l' + y'$. (See Figure 2 where a point lying on a diagonal line labeled by a is greater than a point lying on a diagonal line labeled by b if $a > b$.)

Now define linear operators E_i , F_i , T_i , $i \in I$, and $T_d : \mathcal{F}(\Lambda_k) \rightarrow \mathcal{F}(\Lambda_k)$ as follows,

$$\begin{aligned} E_i &= \sum_{\substack{(l,y) \in \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} \left(\prod_{\substack{(l',y') > (l,y) \\ l'+y' \equiv \pm i \pmod{2n}}} T_{(l',y')}^{+} \right) E_{(l,y)}, \\ F_i &= \sum_{\substack{(l,y) \in \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} \left(\prod_{\substack{(l',y') < (l,y) \\ l'+y' \equiv \pm i \pmod{2n}}} T_{(l',y')}^{-} \right) F_{(l,y)}, \\ T_i^{\pm} &= \prod_{\substack{(l,y) \in \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} T_{(l,y)}^{\pm}, \text{ and} \end{aligned}$$

for $\mathbf{Y} \in \mathcal{Y}(\lambda)$, $T_d(\mathbf{Y}) = q^{-(\text{the number of 0-coloured boxes in } \mathbf{Y})}$

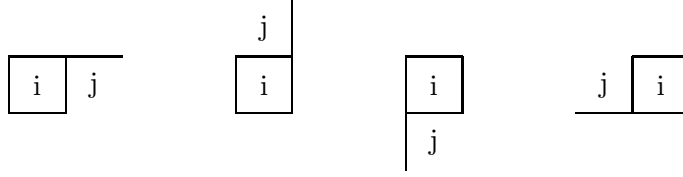


FIGURE 3.

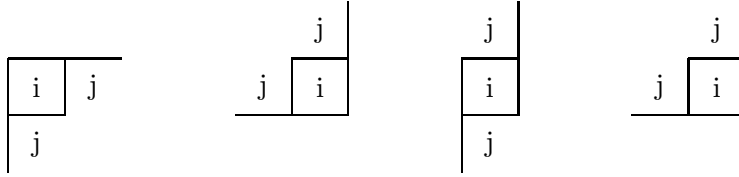


FIGURE 4.

2.3. Theorem. *The vector space $\mathcal{F}(\Lambda_k)$, where $k \in I$, is a $\mathcal{U}_q(\mathfrak{g})$ -module where the action of the generators e_i , f_i , q^{h_i} and q^d is given by that of E_i , F_i , T_i and T_d , respectively.*

Proof. We have to show that E_i , F_i , T_i and T_d satisfy the defining relations of e_i , f_i , q^{h_i} and q^d .

(1) is clear.

(2) For $i, j \in I$, $T_j E_i = q^{a_{ji}} E_i T_j$ will follow from $T_j E_{(l,y)} = q^{a_{ji}} E_{(l,y)} T_j$ for $(l, y) \in \mathbb{N} \times \mathbb{Z}$ with $l + y \equiv \pm 1 \pmod{2n}$.

Note that, for $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$, $T_j(\mathbf{Y}) = q^{ccj(\mathbf{Y}) - cxj(\mathbf{Y})}$, where $ccj(\mathbf{Y}) := \#$ of concave j -coloured corners in \mathbf{Y} and $cxj(\mathbf{Y}) := \#$ of convex j -coloured corners in \mathbf{Y} .

Assume that $E_{(l,y)}(\mathbf{Y}) \neq 0$.

If i and j are not adjacent nodes in the Dynkin diagram and $i \neq j$, then $ccj(\mathbf{Y}) - cxj(\mathbf{Y}) = ccj(E_{(l,y)}(\mathbf{Y})) - cxj(E_{(l,y)}(\mathbf{Y}))$.

If i and j are adjacent nodes in the Dynkin diagram and $i \neq 0$ or n , then $E_i(\mathbf{Y})$ has one less concave j -coloured corner or one more convex j -coloured corner than \mathbf{Y} (Figure 3 shows all the possible cases.) In either case $ccj(E_{(l,y)}(\mathbf{Y})) - cxj(E_{(l,y)}(\mathbf{Y})) = ccj(\mathbf{Y}) - cxj(\mathbf{Y}) - 1$.

If $i = 0$ and $j = 1$ (or $i = n$ and $j = n - 1$), then $E_i(\mathbf{Y})$ has two less concave j -coloured corners or one less concave j -coloured corner and one more convex j -coloured corner or two more convex j -coloured corners than \mathbf{Y} (Figure 4 shows all the possible cases.) In all case $ccj(E_{(l,y)}(\mathbf{Y})) - cxj(E_{(l,y)}(\mathbf{Y})) = ccj(\mathbf{Y}) - cxj(\mathbf{Y}) - 2$.

If $i = j$, $E_i(\mathbf{Y})$ has one less convex i -coloured corner and one more concave i -coloured corner than \mathbf{Y} . Hence $ccj(E_{(l,y)}(\mathbf{Y})) - cxj(E_{(l,y)}(\mathbf{Y})) = ccj(\mathbf{Y}) - cxj(\mathbf{Y}) + 2$.

We now show that $T_d E_i = q^{\delta_{0,i}} E_i T_d$. If $i \neq 0$, T_d and E_i commute. If $i = 0$ and $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$ then either $E_0(\mathbf{Y}) = 0$ or

$T_d E_0(\mathbf{Y}) = q^-$ the # of 0-coloured boxes in $\mathbf{Y}^{+1} E_0(\mathbf{Y})$. In either case the result follows.

The proof of (3) is similar.

(4) For $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$, $i \in I$ and $(l, y) \in \mathbb{N} \times \mathbb{Z}$, define

$$a(i, l, y, \mathbf{Y}) = \#\{(l', y') : (l', y') \text{ is a concave } i - \text{coloured corner in } \mathbf{Y} \text{ and}$$

$$(l', y') > (l, y)\}$$

$$-\#\{(l', y') : (l', y') \text{ is a convex } i - \text{coloured corner in } \mathbf{Y} \text{ and}$$

$$(l', y') > (l, y)\} \text{ and}$$

$$b(i, l, y, \mathbf{Y}) = \#\{(l', y') : (l', y') \text{ is a convex } i - \text{coloured corner in } \mathbf{Y} \text{ and}$$

$$(l', y') < (l, y)\}$$

$$-\#\{(l', y') : (l', y') \text{ is a concave } i - \text{coloured corner in } \mathbf{Y} \text{ and}$$

$$(l', y') < (l, y)\}.$$

Then

$$E_i(\mathbf{Y}) = \sum_{\substack{(l, y) \in \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} q_i^{a(i, l, y, \mathbf{Y})} E_{(l, y)}(\mathbf{Y}),$$

$$F_i(\mathbf{Y}) = \sum_{\substack{(l, y) \in \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} q_i^{b(i, l, y, \mathbf{Y})} F_{(l, y)}(\mathbf{Y}),$$

$$E_i(F_j(\mathbf{Y})) = \sum_{\substack{(l, y) \in \\ \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} \sum_{\substack{(l_1, y_1) \in \\ \mathbb{N} \times \mathbb{Z} \\ l_1+y_1 \equiv \pm j \pmod{2n}}} q_i^{a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y}))} q_j^{b(j, l_1, y_1, \mathbf{Y})} E_{(l, y)}(F_{(l_1, y_1)}(\mathbf{Y})),$$

$$F_j(E_i(\mathbf{Y})) = \sum_{\substack{(l, y) \in \\ \mathbb{N} \times \mathbb{Z} \\ l+y \equiv \pm i \pmod{2n}}} \sum_{\substack{(l_1, y_1) \in \\ \mathbb{N} \times \mathbb{Z} \\ l_1+y_1 \equiv \pm j \pmod{2n}}} q_i^{a(i, l, y, \mathbf{Y})} q_j^{b(j, l_1, y_1, E_{(l_1, y_1)}(\mathbf{Y}))} F_{(l_1, y_1)}(E_{(l, y)}(\mathbf{Y})),$$

In what follows, $(l, y), (l_1, y_1) \in \mathbb{N} \times \mathbb{Z}$ with $l+y \equiv \pm i \pmod{2n}$, $l_1+y_1 \equiv \pm j \pmod{2n}$. Note that $E_{(l, y)} F_{(l_1, y_1)} = F_{(l_1, y_1)} E_{(l, y)}$ unless $(l_1, y_1) = (l-1, y+1)$. We will assume that $E_{(l, y)} F_{(l_1, y_1)}(\mathbf{Y}) \neq 0$ or $F_{(l_1, y_1)} E_{(l, y)}(\mathbf{Y}) \neq 0$.

If i and j are not adjacent in the Dynkin diagram and $i \neq j$, or if $(l_1, y_1) < (l, y)$

$$a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y})) = a(i, l, y, (\mathbf{Y})),$$

$$b(j, l_1, y_1, \mathbf{Y}) = b(j, l_1, y_1, E_{(l,y)}(\mathbf{Y})).$$

If i and j are adjacent and $(l_1, y_1) > (l, y)$, we have three cases.

Case 1. $j = 0$ and $i = 1$ (or $j = n$ and $i = n - 1$).

$$a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y})) = a(i, l, y, (\mathbf{Y})) + 2,$$

$$b(j, l_1, y_1, \mathbf{Y}) = b(j, l_1, y_1, E_{(l,y)}(\mathbf{Y})) - 1.$$

Case 2. $i = 0$ and $j = 1$ (or $i = n$ and $j = n - 1$).

$$a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y})) = a(i, l, y, (\mathbf{Y})) + 1,$$

$$b(j, l_1, y_1, \mathbf{Y}) = b(j, l_1, y_1, E_{(l,y)}(\mathbf{Y})) - 2.$$

Case 3. $j \neq 0, n$, $i \neq 0, n$ and $i \neq j$.

$$a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y})) = a(i, l, y, (\mathbf{Y})) + 1,$$

$$b(j, l_1, y_1, \mathbf{Y}) = b(j, l_1, y_1, E_{(l,y)}(\mathbf{Y})) - 1.$$

Hence if $i \neq j$, $E_i F_j = F_j E_i$.

If $i = j$ and $(l_1, y_1) > (l, y)$,

$$a(i, l, y, F_{(l_1, y_1)}(\mathbf{Y})) = a(i, l, y, (\mathbf{Y})) - 2,$$

$$b(j, l_1, y_1, \mathbf{Y}) = b(j, l_1, y_1, E_{(l,y)}(\mathbf{Y})) + 2.$$

So the only terms in $(E_i F_i - F_i E_i)(\mathbf{Y})$ which give non-zero contributions to the sum are those terms where $(l_1, y_1) = (l - 1, y + 1)$, and $E_{(l,y)}(F_{(l_1, y_1)}(\mathbf{Y})) \neq 0$ or $F_{(l_1, y_1)}(E_{(l,y)}(\mathbf{Y})) \neq 0$.

A concave corner in \mathbf{Y} at site (l, y) will contribute $q_i^{a(i, l, y, F_{(l,y)}(\mathbf{Y}))b(i, l, y, \mathbf{Y})} \mathbf{Y}$ to $(E_i F_i - F_i E_i)(\mathbf{Y})$ and a convex corner in \mathbf{Y} at site (l, y) a $-q_i^{a(i, l, y, \mathbf{Y})b(i, l, y, E_{(l,y)}(\mathbf{Y}))} \mathbf{Y}$.

Let $(l_1, y_1) > (l_2, y_2) > \dots > (r_l, k_l, y_l)$ be the sites of the i -coloured corners of \mathbf{Y} , σ be the i -signature of \mathbf{Y} (see 3.2) and $J(\sigma)$ be defined as in section 3.2. Then the contribution to $(E_i F_i - F_i E_i)(\mathbf{Y})$ of the corners in \mathbf{Y} corresponding to $\{1, 2, \dots, l\} \setminus J(\sigma)$ cancel out. Let $a = \#$ of 1's in $J(\sigma)$ and $b = \#$ of 0's in $J(\sigma)$. then

$$(E_i F_i - F_i E_i)(\mathbf{Y}) = \frac{q_i^{b-a} - q_i^{-(b-a)}}{q_i - q_i^{-1}} \mathbf{Y} = \frac{T_i^+ - T_i^-}{q_i - q_i^{-1}} \mathbf{Y}$$

(5) If $a_{ij} = 0$ i.e. i and j are not adjacent, $E_i E_j = E_j E_i$.

If $a_{ij} = -1$, and $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$,

$$E_i^2 E_j(\mathbf{Y}) = \sum_{\mathbf{X}} a(\mathbf{X}) \mathbf{X},$$

$$E_i E_j E_i(\mathbf{Y}) = \sum_{\mathbf{X}} b(\mathbf{X}) \mathbf{X},$$

$$E_j E_i^2(\mathbf{Y}) = \sum_{\mathbf{X}} c(\mathbf{X}) \mathbf{X},$$

where $a(\mathbf{X}), b(\mathbf{X})$ and $c(\mathbf{X}) \in \mathbb{Q}(q)$ and the sums run over all $\mathbf{X} \in \mathcal{Y}(\Lambda_k)$ which are obtained from $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$ by removing two i -coloured boxes and one j -coloured box.

Let $(l_1, y_1), (l_2, y_2)$ and (l_3, y_3) be the co-ordinates of the bottom right corners of two distinct i -coloured boxes and one j -coloured box of \mathbf{Y} (if they exist), resp., and let \mathbf{X} be obtained from \mathbf{Y} by removing these three boxes. Assume $\mathbf{X} \in \mathcal{Y}(\Lambda_k)$.

Case1. If $(l_1, y_1), (l_2, y_2)$ and (l_3, y_3) are sites of two i -coloured convex corners and one j -coloured convex corner of \mathbf{Y} , we consider three cases.

Case 1 (a). $(l_1, y_1), (l_2, y_2) > (l_3, y_3)$.

$$a(\mathbf{X}) = q^m ([2]_{q_i}),$$

$$b(\mathbf{X}) = q^m q_j^{-a_{ji}} ([2]_{q_i}) = q^m q_i^{-1} ([2]_{q_i}), \text{ and}$$

$$c(\mathbf{X}) = q^m q_j^{-2a_{ji}} ([2]_{q_i}) = q^m q_i^{-2} ([2]_{q_i}), \text{ for some } m \in \mathbb{Z}.$$

Case 1 (b). $(l_1, y_1) > (l_3, y_3) > (l_2, y_2)$.

$$a(\mathbf{X}) = q^m q_i^{-1} ([2]_{q_i}),$$

$$b(\mathbf{X}) = q^m (q_j^{-a_{ji}} q_i^{-1} q_i + q_i^{-1}) = q^m (2q_i^{-1}),$$

$$c(\mathbf{X}) = q^m q_j^{-a_{ji}} ([2]_{q_i}) = q^m q_i^{-1} ([2]_{q_i}), \text{ for some } m \in \mathbb{Z}.$$

Case 1 (c). $(l_3, y_3) > (l_1, y_1), (l_2, y_2)$.

$$a(\mathbf{X}) = q^m q_i^{-2} ([2]_{q_i}),$$

$$b(\mathbf{X}) = q^m q_i^{-1} ([2]_{q_i}),$$

$$c(\mathbf{X}) = q^m([2]_{q_i}), \text{ for some } m \in \mathbb{Z}.$$

Case2. If the j -coloured box corresponding to (l_3, y_3) is **hidden** by (i.e. if it is immediately to the left or above of) the i -coloured box corresponding to (l_2, y_2) , we consider two cases.

Case 2 (a). $(l_1, y_1) > (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = q^m q_i^{-1}, \text{ and}$$

$$c(\mathbf{X}) = q^m q_j^{-a_{ji}}([2]_{q_i}) = q^m q_i^{-1}([2]_{q_i}), \text{ for some } m \in \mathbb{Z}.$$

Case 2 (b). $(l_1, y_1) < (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = q^m,$$

$$c(\mathbf{X}) = q^m [2]_{q_i}, \text{ for some } m \in \mathbb{Z}.$$

Case3. If the i -coloured box corresponding to (l_2, y_2) is hidden by the j -coloured box corresponding to (l_3, y_3) , we consider two cases.

Case 3 (a). $(l_1, y_1) > (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = q^m [2]_{q_i},$$

$$b(\mathbf{X}) = q^m q_j^{-a_{ji}} q_i = 1, \text{ and}$$

$$c(\mathbf{X}) = 0, \text{ for some } m \in \mathbb{Z}.$$

Case 3 (b). $(l_1, y_1) < (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = q^m [2]_{q_i},$$

$$b(\mathbf{X}) = q^m,$$

$$c(\mathbf{X}) = 0, \text{ for some } m \in \mathbb{Z}.$$

In all cases $a(\mathbf{X}) - [2]_{q_i} b(\mathbf{X}) + c(\mathbf{X}) = 0$.

If $a_{ij} = -2$, and $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$,

$$\begin{aligned} E_i^3 E_j(\mathbf{Y}) &= \sum_{\mathbf{X}} a(\mathbf{X}) \mathbf{X}, \\ E_i^2 E_j E_i(\mathbf{Y}) &= \sum_{\mathbf{X}} b(\mathbf{X}) \mathbf{X}, \\ E_i E_j E_i^2(\mathbf{Y}) &= \sum_{\mathbf{X}} c(\mathbf{X}) \mathbf{X}, \\ E_j E_i^3(\mathbf{Y}) &= \sum_{\mathbf{X}} d(\mathbf{X}) \mathbf{X}, \end{aligned}$$

where $a(\mathbf{X}), b(\mathbf{X}), c(\mathbf{X})$ and $d(\mathbf{X}) \in \mathbb{Q}(q)$ and the sums run over all $\mathbf{X} \in \mathcal{Y}(\Lambda_k)$ which are obtained from \mathbf{Y} by removing three i -coloured boxes and one j -coloured box.

Let $(l_1, y_1), (l_2, y_2), (l_3, y_3)$ and (l_4, y_4) be the co-ordinates of the bottom right corners of three distinct i -coloured boxes and one j -coloured box of \mathbf{Y} (if they exist), resp., and let \mathbf{X} be obtained from \mathbf{Y} by removing these four boxes. Assume $\mathbf{X} \in \mathcal{Y}(\Lambda_k)$.

Case 1. If \mathbf{Y} has three convex i -coloured corners and one convex j -coloured corner at sites $(l_1, y_1), (l_2, y_2), (l_3, y_3)$ and (l_4, y_4) , we consider four cases.

Case 1 (a). $(l_1, y_1), (l_2, y_2), (l_3, y_3) > (l_4, y_4)$.

$$\begin{aligned} a(\mathbf{X}) &= q^m ([3]_q [2]_q), \\ b(\mathbf{X}) &= q^m q^{-2} ([3]_q [2]_q), \\ c(\mathbf{X}) &= q^m q^{-4} ([3]_q [2]_q), \\ d(\mathbf{X}) &= q^m q^{-6} ([3]_q [2]_q), \text{ for some } m \in \mathbb{Z}. \end{aligned}$$

Case 1 (b). $(l_1, y_1), (l_2, y_2) > (l_4, y_4) > (l_3, y_3)$.

$$\begin{aligned} a(\mathbf{X}) &= q^m q^{-2} ([3]_q [2]_q), \\ b(\mathbf{X}) &= q^m (2q^{-1} + 3q^{-3} + q^{-5}), \end{aligned}$$

$$c(\mathbf{X}) = q^m(q^{-1} + 3q^{-3} + 2q^{-5}),$$

$$d(\mathbf{X}) = q^m q^{-4}([3]_q[2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 1 (c). $(l_1, y_1) > (l_4, y_4) > (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = q^m q^{-4}([3]_q[2]_q),$$

$$b(\mathbf{X}) = q^m(q^{-1} + 3q^{-3} + 2q^{-5}),$$

$$c(\mathbf{X}) = q^m(2q^{-1} + 3q^{-3} + q^{-5}),$$

$$d(\mathbf{X}) = q^m q^{-2}([3]_q[2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 1 (d). $(l_4, y_4) > (l_1, y_1), (l_2, y_2), (l_3, y_3)$.

$$a(\mathbf{X}) = q^m q^{-6}([3]_q[2]_q),$$

$$b(\mathbf{X}) = q^m q^{-4}([3]_q[2]_q),$$

$$c(\mathbf{X}) = q^m q^{-2}([3]_q[2]_q),$$

$$d(\mathbf{X}) = q^m([3]_q[2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 2. If \mathbf{Y} has three convex i -coloured corners at sites (l_1, y_1) , (l_2, y_2) , and (l_3, y_3) and the j -coloured box corresponding to (l_4, y_4) is hidden by that corresponding to (l_3, y_3) but not by the other i -coloured boxes, we consider three cases.

Case 2 (a). $(l_1, y_1), (l_2, y_2) > (l_3, y_3), (l_4, y_4)$.

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = q^m q^{-2}[2]_q,$$

$$c(\mathbf{X}) = q^m(q^{-4} + q^{-2})[2]_q,$$

$$d(\mathbf{X}) = q^m q^{-4}([3]_q[2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 2 (b). $(l_1, y_1) > (l_3, y_3), (l_4, y_4) > (l_2, y_2)$.

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = q^m q^{-1} [2]_q,$$

$$c(\mathbf{X}) = q^m (q^{-1} + q^{-2}) [2]_q,$$

$$d(\mathbf{X}) = q^m q^{-2} ([3]_q [2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 2 (c). $(l_3, y_3), (l_4, y_4) > (l_1, y_1), (l_2, y_2), .$

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = q^m q^{-2} [2]_q,$$

$$c(\mathbf{X}) = q^m (1 + q^{-2}) [2]_q,$$

$$d(\mathbf{X}) = q^m ([3]_q [2]_q), \text{ for some } m \in \mathbb{Z}.$$

Case 3. If \mathbf{Y} has three convex i -coloured corners at sites (l_1, y_1) , (l_2, y_2) , and (l_3, y_3) and the j -coloured box corresponding to (l_4, y_4) is hidden by those corresponding to (l_2, y_2) and (l_3, y_3) , then

$$a(\mathbf{X}) = 0,$$

$$b(\mathbf{X}) = 0,$$

$$c(\mathbf{X}) = q^m [2]_q$$

$$d(\mathbf{X}) = q^m [3]_q [2]_q, \text{ for some } m \in \mathbb{Z}.$$

Case 4. If \mathbf{Y} has two convex i -coloured corners at sites (l_1, y_1) and (l_2, y_2) and one j -coloured corner at site (l_4, y_4) , and the i -coloured box corresponding to (l_3, y_3) is hidden by that corresponding to (l_4, y_4) , we consider three cases.

Case 4 (a). $(l_1, y_1), (l_2, y_2) > (l_3, y_3), (l_4, y_4)$.

$$a(\mathbf{X}) = q^m [3]_q [2]_q,$$

$$\begin{aligned}
b(\mathbf{X}) &= q^m(1 + q^{-2})[2]_q, \\
c(\mathbf{X}) &= q^m q^{-2}[2]_q, \\
d(\mathbf{X}) &= 0, \text{ for some } m \in \mathbb{Z}.
\end{aligned}$$

Case 4 (b). $(l_1, y_1) > (l_3, y_3), (l_4, y_4) > (l_2, y_2)$.

$$\begin{aligned}
a(\mathbf{X}) &= q^m q^{-1}[3]_q[2]_q, \\
b(\mathbf{X}) &= q^m(2q^{-1})[2]_q, \\
c(\mathbf{X}) &= q^m q^{-1}[2]_q, \\
d(\mathbf{X}) &= 0, \text{ for some } m \in \mathbb{Z}.
\end{aligned}$$

Case 4 (c). $(l_3, y_3), (l_4, y_4) > (l_1, y_1), (l_2, y_2), .$

$$\begin{aligned}
a(\mathbf{X}) &= q^m q^{-2}[3]_q[2]_q, \\
b(\mathbf{X}) &= q^m(1 + q^{-2})[2]_q, \\
c(\mathbf{X}) &= q^m[2]_q, \\
d(\mathbf{X}) &= 0, \text{ for some } m \in \mathbb{Z}.
\end{aligned}$$

Case 5. If \mathbf{Y} has one convex i -coloured corners at site (l_1, y_1) , one convex j -coloured corner at site (l_4, y_4) , and the i -coloured boxes corresponding to (l_2, y_2) and (l_3, y_3) are hidden by that corresponding to (l_4, y_4) , then

$$\begin{aligned}
a(\mathbf{X}) &= q^m[3]_q[2]_q, \\
b(\mathbf{X}) &= q^m[2]_q, \\
c(\mathbf{X}) &= 0 \\
d(\mathbf{X}) &= 0, \text{ for some } m \in \mathbb{Z}.
\end{aligned}$$

Case 6. If \mathbf{Y} has two convex i -coloured corners at sites (l_1, y_1) and (l_2, y_2) , and the j -coloured box corresponding to (l_4, y_4) is hidden by that corresponding to (l_2, y_2) and the i -coloured box corresponding to (l_3, y_3) is hidden by that corresponding to (l_4, y_4) then

$$\begin{aligned} a(\mathbf{X}) &= 0, \\ b(\mathbf{X}) &= q^m [2]_q, \\ c(\mathbf{X}) &= q^m [2]_q, \\ d(\mathbf{X}) &= 0, \text{ for some } m \in \mathbb{Z}. \end{aligned}$$

In all cases $a(\mathbf{X}) - [3]_{q_i} b(\mathbf{X}) + [3]_{q_i} c(\mathbf{X}) - d(\mathbf{X}) = 0$.

(6) is similar to (5). □

2.4. Lemma. *The $\mathcal{U}_q(\mathfrak{g})$ -module $\mathcal{F}(\Lambda_k)$ belongs to the category \mathcal{O}_{int} .*

Proof. Note that $q^h \phi_k = q^{\Lambda_k(h)} \phi_k$ for all $h \in P^\vee$. By induction on the number of boxes of an element $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$, we can show that $q^h \mathbf{Y} = q^{\mu(h)} \mathbf{Y}$ for all $h \in P^\vee$, where $\mu = \Lambda_k - \sum_{i=0}^n k_i \alpha_i$ and $k_i = \#$ of i -coloured boxes in \mathbf{Y} . □

2.5. Corollary. *$M(\Lambda_k) := \mathcal{U}_q(\mathfrak{g}) \phi_k$ is the irreducible integrable highest weight module of highest weight Λ_k .*

THE CRYSTAL BASE FOR $\mathcal{F}(\Lambda_k)$ AND THE CRYSTAL $B(\Lambda_k)$

The proofs of the following two theorems are as in [2] and [10].

3.1. Theorem. *Let $A = \{\frac{f(q)}{g(q)} : f(q), g(q) \in \mathbb{Q}[q] \text{ and } g(0) \neq 0\}, k \in I$, $L(\mathcal{F}(\Lambda_k)) = \sum_{\mathbf{Y} \in \mathcal{F}(\Lambda_k)} A \mathbf{Y}$ and $B(\mathcal{F}(\Lambda_k)) = \mathcal{Y}(\Lambda_k)$, where we identify $\mathbf{Y} + qL(\mathcal{F}(\Lambda_k))$ with \mathbf{Y} . Then $(L(\mathcal{F}(\Lambda_k)), B(\mathcal{F}(\Lambda_k)))$ is an (upper) crystal base for the integrable $\mathcal{U}_q(\mathfrak{g})$ -module $\mathcal{F}(\Lambda_k)$.*

Note. If we replace the definition of E_i and F_i by

$$E_i = \sum_{\substack{(r,y) \in \mathbb{N} \times \mathbb{Z} \\ k+y \equiv \pm i \pmod{2n}}} \left(\prod_{\substack{(k_1, y_1) < (r,y) \\ k_1+y_1 \equiv \pm i \pmod{2n}}} T_{(k_1, y_1)}^- \right) E_{(r,y)},$$

$$F_i = \sum_{\substack{(r,y) \in \mathbb{N} \times \mathbb{Z} \\ k+y \equiv \pm i \pmod{2n}}} \left(\prod_{\substack{(k_1, y_1) > (r, y) \\ k_1 + y_1 \equiv \pm i \pmod{2n}}} T_{(k_1, y_1)}^+ \right) F_{(r, y)},$$

the pair $(L(\mathcal{F}(\Lambda_k)), B(\mathcal{F}(\Lambda_k)))$ is a (lower) crystal base of $\mathcal{F}(\Lambda_k)$.

3.2. Definition. Let $\mathbf{Y} \in \mathcal{Y}(\Lambda_k)$ and $(k_1, y_1) > (k_2, y_2) > \cdots > (k_l, y_l)$ be the sites of the i -coloured corners of \mathbf{Y} and, define the i -signature of \mathbf{Y} to be the l -tuple $\sigma = (\sigma_1, \dots, \sigma_l)$, where for $1 \leq m \leq l$,

$$\sigma_m := \begin{cases} 0 & \text{if there is a concave corner in } \mathbf{Y} \text{ at site } (k_m, y_m) \\ 1 & \text{if there is a convex corner in } \mathbf{Y} \text{ at site } (k_m, y_m). \end{cases}$$

Define $J(\sigma)$ as follows: let $J = \{1, \dots, m\}$.

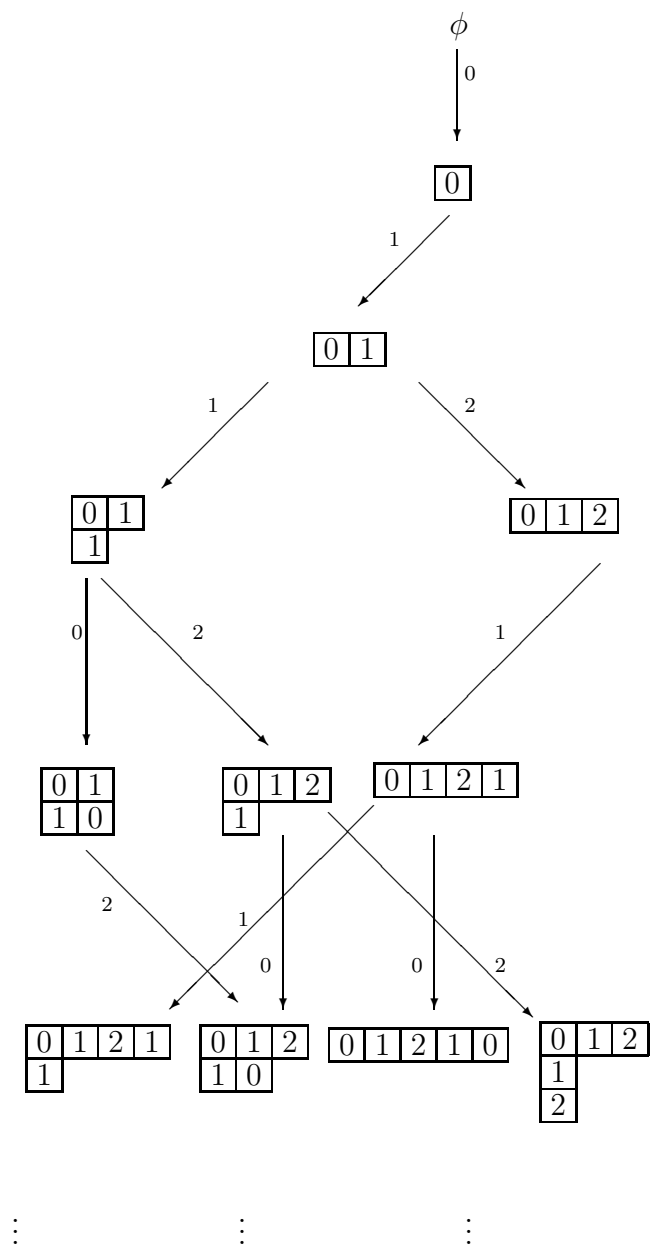
- (i) If there exists $r < s$ such that $(\sigma_r, \sigma_s) = (0, 1)$ and $r' \notin J$ for $r < r' < s$, replace J by $J \setminus \{r, s\}$ and repeat this step;
- (ii) otherwise let $J(\sigma) = J$.

If there exists an $i_r \in J(\sigma)$ with $\sigma_{i_r} = 1$, define $\tilde{e}_i(\mathbf{Y})$ to be the same as \mathbf{Y} with the i -coloured convex corner corresponding to the largest such i_r removed; otherwise, define $\tilde{e}_i(\mathbf{Y})$ to be zero.

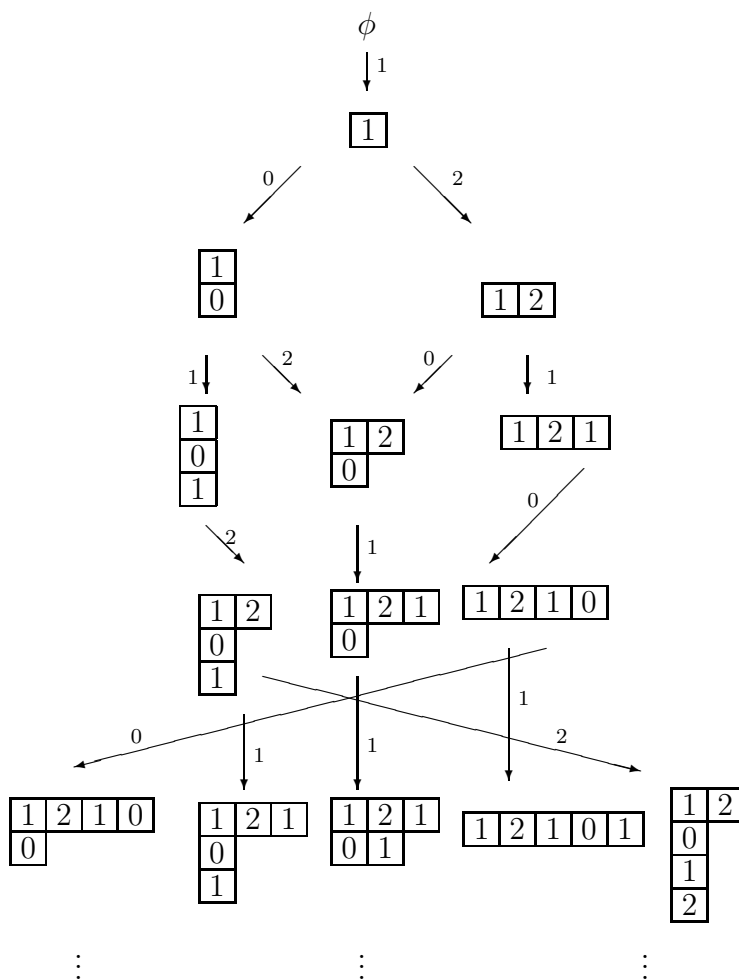
If there exists an $i_r \in J(\sigma)$ with $\sigma_{i_r} = 0$, define $\tilde{f}_i(\mathbf{Y})$ to be the same as \mathbf{Y} with the i -coloured concave corner corresponding to the smallest such i_r replaced by an i -coloured convex corner; otherwise, define $\tilde{f}_i(\mathbf{Y})$ to be zero.

3.3. Theorem. The operators \tilde{e}_i and \tilde{f}_i defined above coincide with the Kashiwara's operators (see [9] for the definition of the Kashiwara's operators).

The crystal graph $B(\Lambda_0)$ for $C_2^{(1)}$.



The crystal graph $B(\Lambda_1)$ for $C_2^{(1)}$.



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